

Stability of MOTS in totally geodesic null horizons

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Abstract

Closed sections of totally geodesic null hypersurfaces are marginally outer trapped surfaces (MOTS), for which a well-defined notion of stability exists. In this paper we obtain the explicit form for the stability operator for such MOTS and analyze in detail its properties in the particular case of non-evolving horizons, which include both isolated and Killing horizons. We link these stability properties with the surface gravity of the horizon and/or to the existence of minimal sections. The results are used, in particular, to obtain an area-angular momentum inequality for sections of axially symmetric horizons in four spacetime dimensions, which helps clarifying the relationship between two different approaches to this inequality existing in the literature.

1 Introduction

Horizons play an important role in gravity theory and several different types arise in many contexts. Particularly relevant are Killing horizons, which are defined in spacetimes admitting a Killing vector ξ as null hypersurfaces \mathcal{H}_ξ where the Killing field ξ is null, nowhere-zero and tangent to \mathcal{H}_ξ . An immediate consequence of this definition is that any closed (i.e. compact and without boundary) spacelike section of a Killing horizon is a marginally outer trapped surface (MOTS for short), namely a closed codimension-two surface with one of its null expansions identically vanishing. MOTS are very interesting objects both from a physical viewpoint, as quasi-local replacements of black holes, and from a mathematical point of view, as objects sharing several important properties with minimal hypersurfaces. In particular, MOTS admit a sensible definition of stability, closely related to the existence of outer trapped surfaces just inside the MOTS. Since Killing horizons are foliated by MOTS, it is natural to link the stability notion of its sections with the geometry of the Killing horizon itself.

However, Killing horizons are unnecessarily restrictive in the sense that their definition requires the existence of a special vector field in the spacetime. The definition of Killing

horizon can be relaxed substantially by extracting the fundamental geometric properties of \mathcal{H}_ξ which follow from the existence of a Killing vector and imposing them directly on the null hypersurface. This has lead to the introduction of so-called non-expanding horizons, and their particularizations of weakly isolated horizons and isolated horizons, which have been extensively studied in the literature. *Non-expanding horizons* [21, 22, 9, 5, 6, 7, 20, 28] are embedded null hypersurfaces \mathcal{H} (usually with an additional topological restriction) with vanishing null expansion and such that the Einstein tensor on \mathcal{H} satisfies an appropriate energy condition, which implies, among other things that the second fundamental form of \mathcal{H} vanishes. This, in turn, implies that the null hypersurface admits a canonical connection \mathcal{D} inherited from the spacetime.

To be more precise, define a null normal to \mathcal{H} as a vector field ℓ which is null, nowhere zero and tangent to \mathcal{H} . Null normals ℓ are obviously defined up to an arbitrary nowhere-zero rescaling. The energy condition required in the definition of non-expanding horizon is $\text{Ein}(\ell, u)|_{\mathcal{H}} \geq 0$ for any causal vector u with the same time-orientation than ℓ (Ein the Einstein tensor of the spacetime).

Equivalent classes $[\ell]$ of null normals are defined by the equivalence relation $\{\ell' \sim \ell$ if and only if $\ell' = c\ell\}$ with c a nonzero constant. A *weakly isolated horizon* [7] is a non-expanding horizon with a selected class of null normals $[\ell]$ such that the commutator of the Lie derivative \mathcal{L}_ℓ and the covariant derivative \mathcal{D} satisfies $[\mathcal{D}, \mathcal{L}_\ell]\ell = 0$, for any $\ell \in [\ell]$. This property, in combination with the energy condition above, implies that the surface gravity along ℓ , (i.e. the function $\kappa_\ell : \mathcal{H} \rightarrow \mathbb{R}$ defined by $\nabla_\ell \ell = \kappa_\ell \ell$) is constant. *Isolated horizons* [9] \mathcal{H} are weakly isolated horizon with the additional property that $[\mathcal{D}, \mathcal{L}_\ell] = 0$, for any $\ell \in [\ell]$. Killing horizons \mathcal{H}_ξ are isolated horizons whenever the class of null normals $[\ell]$ is chosen to be $[\xi]$ and provided the spacetime satisfies the energy condition mentioned before. Non-expanding, weakly isolated and isolated horizons were first introduced and studied in four spacetime dimensions, but all the definitions and most of the results carry over to arbitrary spacetime dimension [35, 34].

Although the energy condition imposed on non-expanding horizons is physically reasonable, in geometric terms it is perhaps more natural to impose conditions directly on the geometry of \mathcal{H} irrespectively of the energy-contents of the spacetime. As already mentioned, the main consequence of the energy condition for non-expanding horizon is that the null hypersurface is totally geodesic. It thus becomes of interest to analyze the geometry of such hypersurfaces (this was in fact the approach taken in the seminal paper by P. Hájíček [21]). In this context, isolated horizons are naturally replaced by totally geodesic null hypersurfaces with a selected null normal ℓ satisfying $[\mathcal{D}, \mathcal{L}_\ell] = 0$. Although closely related to isolated horizons, this type of hypersurfaces is clearly more general. We call them *non-evolving horizons* in this paper.

Similarly as for Killing horizons, any closed spacelike section of a totally geodesic null hypersurface is a MOTS. It is therefore of interest to try and relate the stability properties of those MOTS with the geometry of the null hypersurface. Isolated horizons admit several notions of extremality [11]. One of them involves the existence of trapped surfaces just inside the horizon. This type of horizons are called subextremal, and they have played an important role in the proof of area-angular momentum inequalities [24] in the case of

axially symmetric horizons in four spacetime dimensions. The subextremality condition of horizons is closely related to the stability of the MOTS embedded in the horizon, so studying in detail the stability of these sections also clarifies under which conditions an isolated horizon is subextremal or not.

Our aim in this paper is to perform a detailed study of the stability of MOTS in totally geodesic null hypersurfaces and, particularly, in non-evolving horizons. We obtain a simple and explicit form for the stability operator of MOTS lying in totally geodesic null horizons (Proposition 2 and Corollary 2). As a direct consequence we find (Corollary 3) that non-evolving horizons with vanishing surface gravity are always marginally stable. We also prove (Proposition 3) a result that relates the stability properties of the sections with the sign of the surface gravity of the horizon, provided the MOTS is a marginally trapped surface. This extends previous results obtained by Booth and Fairhurst [11] in the case of spacetime dimension four and axially symmetric isolated horizons. We also study the dependence of the stability properties with the choice of section in the horizon. We find that, in the constant surface gravity case, the stability operator transforms nicely with the change of section and that the stability properties are independent of the section. However, perhaps contrarily to our intuition, the same is not true in the case of non-constant surface gravity. We give in Lemma 4 an explicit example of a Killing horizon with non-constant surface gravity where the stability depends on the choice of section.

One of our main results (Theorem 1) states that non-evolving horizons with non-zero and constant surface gravity are marginally stable if and only if they admit a section with vanishing total null expansion. Under suitable additional conditions (which are automatically satisfied in static Killing horizons or in axially symmetric non-evolving horizons of spherical topology) this section must, in fact, be minimal. An interesting consequence of the results in this paper is that they allow us to improve our understanding of the relationship between the area-angular momentum inequality proved by Hennig, Ansorg and Cederbaum [24] in the context of stationary and axially symmetric four dimensional spacetimes and the local proof obtained by Jaramillo, Reiris and Dain [32] for stable MOTS. Some initial insight on the relationship between the two approaches has been recently obtained by Jaramillo and Gabach-Clément [19] by showing that the key integral consequences of “subextremality” and “strict stability” employed respectively in [25] and [32] (in the context of an area-angular momentum-charge inequality) translate exactly into one another under a suitable renaming of functions. Recently a clarification of the relationship between the inequality of Hennig *et al.* [24] and another area-angular momentum inequality due to Dain and Reiris [17] and valid for minimal surfaces lying in maximal spacelike hypersurfaces in vacuum has also been obtained [15] by working in suitable coordinate systems. Our approach here shows, in a purely geometric and coordinate independent manner, how the framework for the general local inequality for stable MOTS in [32] relates to the framework for the inequality in stationary and axially symmetric black hole horizons [24]

The plan of the paper is as follows. In Section 2 we introduce our notation and basic definitions and recall the concept of stability operator for MOTS. In Section 3 we obtain the explicit form of the stability operator for sections of totally geodesic null hypersurfaces. Section 4 is devoted to analyzing the stability properties of sections of non-evolving hori-

zons. Finally in Section 5 we apply our results to the problem of area-angular momentum inequalities for axially symmetric horizons in four spacetime dimensions. In Theorem 3 we find conditions, well-adapted to the horizon setting, under which the area-angular momentum inequality holds. This result is then used to clarify the relationship between the results of Hennig *et al.* and of Jaramillo *et al.* mentioned above. We finish the paper with a comment on the role played by the area-angular momentum inequality in the proof of non-existence of two-black hole configurations in equilibrium.

2 Notation and basic results

Throughout this paper, a spacetime $(M, g^{(n+1)})$ is an $(n+1)$ -dimensional oriented manifold M together with a smooth metric $g^{(n+1)}$ of Lorentzian signature. We assume $(M, g^{(n+1)})$ to be time-oriented. All manifolds in this paper are connected and without boundary and all geometric objects are assumed to be smooth. Scalar product with a metric h is denoted by $\langle \cdot, \cdot \rangle_h$ except for the spacetime metric, where we simply write $\langle \cdot, \cdot \rangle$. The covariant derivative of a metric h is ∇^h and the corresponding Riemann tensor is denoted by Riem_h (our sign conventions are $\text{Riem}_h(X, Y)Z \stackrel{\text{def}}{=} (\nabla_X^h \nabla_Y^h Z - \nabla_Y^h \nabla_X^h Z - \nabla_{[X, Y]}^h Z)$). The Ricci, Einstein and curvature scalar of Riem_h are denoted by Ric_h , Ein_h and Scal_h (with the sign convention that the Ricci tensor and curvature scalar are positive for a standard sphere). For the spacetime metric $g^{(n+1)}$ we write ∇ , Riem , Ric , Ein and Scal . A spacetime is said to satisfy the **dominant energy condition** if $-\text{Ein}(u, \cdot)$ is causal and future directed for any future causal vector u . Note that this condition, when evaluated on \mathcal{H} with $u = \ell$ is precisely the energy condition imposed in the definition of isolated horizon. Spacetime tensors carry Greek indices, which are lowered and raised with $g^{(n+1)}$. Index components of the Riemann tensor are written as $R^\alpha_{\beta\gamma\delta}$ and are defined by $R^\alpha_{\beta\gamma\delta} \stackrel{\text{def}}{=} (\text{Riem}(e_\gamma, e_\delta)e_\beta)^\alpha$.

In this paper we use the term “spacelike surface” to denote a closed (i.e. compact and without boundary) spacelike, orientable, codimension-two embedded submanifold of $(M, g^{(n+1)})$. We will denote by Φ_S the embedding of S into M and we will often identify S and its image in M under this embedding. The induced metric in S is denoted by h and our convention for the second fundamental form and mean curvature are $\chi(X, Y) \stackrel{\text{def}}{=} -(\nabla_X Y)^\perp$ and $H \stackrel{\text{def}}{=} \text{tr}_h \chi$, where a vector $u \in T_p M, p \in M$, is decomposed as $u = u^\parallel + u^\perp$ according to the direct sum decomposition $T_p M = T_p S \oplus N_p S$ of tangent and normal spaces to S . The second fundamental form along a normal n is defined as $\chi_n \stackrel{\text{def}}{=} \langle \chi, n \rangle$. Its trace defines the null expansion along n , $\theta_n \stackrel{\text{def}}{=} \text{tr}_h \chi_n$. The normal bundle of S , $NS = \bigcup_{p \in S} N_p S$ admits a global basis on null vectors $\{\ell, k\}$ which we always take future-directed and partially normalized by $\langle \ell, k \rangle = -2$ (this leaves the usual boost freedom $\ell' = F\ell, k' = F^{-1}k$, where F is a positive function on S). Given a null basis $\{\ell, k\}$, the connection one-form of the normal bundle is denoted by s_ℓ and defined as

$$s_\ell(X) \stackrel{\text{def}}{=} -\frac{1}{2} \langle k, \nabla_X \ell \rangle.$$

An important notion for spacelike surfaces is the first variation of its null expansions θ_ℓ and θ_k . An explicit form for this variation has been obtained by several authors [38,

23, 13, 20, 10, 3]. In most cases, the derivation uses implicitly that the variation vector ζ is nowhere zero on S . This implies that, given any extension of ζ (which we may assume to be compactly supported without loss of generality) to a neighbourhood of S and the corresponding local one-parameter family of diffeomorphism φ_τ , $\tau \in I_\epsilon \stackrel{\text{def}}{=} (-\epsilon, \epsilon)$, we have the property (after restricting ϵ if necessary) that the map $\Psi_\zeta : S \times I_\epsilon \rightarrow M$ defined by $\Psi_\zeta(p, \tau) \stackrel{\text{def}}{=} \varphi_\tau \circ \Phi_S(p)$ is an embedding. This assumption simplifies notably the calculation of the variation because one can work with Lie derivatives applied to spacetime tensors (see, however, [3] for a derivation which does not make this implicit assumption). In this paper, we only need the variation formula when the variation vector ζ is nowhere zero. Let $\ell^{(\zeta)}$ be a nowhere zero, future-directed null vector on the hypersurface $\Sigma_\zeta \stackrel{\text{def}}{=} \Psi_\zeta(S \times I_\epsilon)$ with the property of being orthogonal to all surfaces $S_\tau \stackrel{\text{def}}{=} \varphi_\tau(S)$, $\tau \in (\epsilon, \epsilon)$. A superindex (ζ) is added because the vector field $\ell^{(\zeta)}$ obviously depends on the variation vector ζ . Note that, given ζ , the field $\ell^{(\zeta)}$ is defined up to a rescaling $\ell^{(\zeta)} \rightarrow Q\ell^{(\zeta)}$, where Q is a positive function on Σ_ζ . Given ζ and $\ell^{(\zeta)}$, the first order variation of θ_ℓ is defined as $\delta_\zeta \theta_\ell \stackrel{\text{def}}{=} \frac{d}{d\tau} \varphi^\star(\theta_{\ell_\tau})|_{\tau=0}$, where θ_{ℓ_τ} is the null expansion of S_τ with respect to the null normal $\ell^{(\zeta)}|_{S_\tau}$. This variation takes the explicit form

$$\begin{aligned} \delta_\zeta \theta_\ell = & -\Delta_h \psi + 2\mathbf{s}_\ell(\nabla^h \psi) + \psi \left(\text{div}_h \mathbf{s}_\ell - \|\mathbf{s}_\ell\|_h^2 + \frac{1}{2}\theta_\ell \theta_k + \frac{1}{2}\text{Scal}_h - \frac{1}{2}\text{Ein}(\ell, k) \right) - \\ & - \alpha(\text{Ein}(\ell, \ell) + \|\chi_\ell\|_h^2) + \kappa_\zeta \theta_\ell, \end{aligned} \quad (1)$$

where α, ψ are defined by the decomposition $\zeta|_S = \alpha\ell - \frac{\psi}{2}k$ and $\kappa_\zeta \stackrel{\text{def}}{=} -\frac{1}{2}\langle k, \nabla_\zeta \ell^{(\zeta)} \rangle|_S$. In this expression Δ_h is the Laplacian on (S, h) , div_h is the divergence with the metric h and $\nabla^h \psi$ is the gradient of ψ . Interchanging $\{\ell, k\}$ in (1) yields [10].

$$\begin{aligned} \delta_\zeta \theta_k = & -\Delta_h \psi - 2\mathbf{s}_\ell(\nabla^h \psi) + \psi \left(-\text{div}_h \mathbf{s}_\ell - \|\mathbf{s}_\ell\|_h^2 + \frac{1}{2}\theta_\ell \theta_k + \frac{1}{2}\text{Scal}_h - \frac{1}{2}\text{Ein}(\ell, k) \right) - \\ & - \alpha(\text{Ein}(k, k) + \|\chi_k\|_h^2) - \kappa_\zeta \theta_k, \end{aligned} \quad (2)$$

where α, ψ are now defined by $\zeta|_S = \alpha k - \frac{\psi}{2}\ell$ (and both κ_ζ and \mathbf{s}_ℓ are the same as in (1)).

These expressions are well-suited for situations where one has good control on the geometric properties of S . However, in some cases one has better knowledge of the properties of the variation vector ζ . An appropriate expression for the variation of the null expansion in this setting can be obtained in terms of the so-called, **deformation tensor** of ζ , defined as $a^\zeta \equiv \mathcal{L}_\zeta g^{(n+1)}$, where \mathcal{L} denotes Lie derivative. The following proposition was proved in [14] with the aim of analyzing the interplay between MOTS and symmetries (given two 2-covariant tensors A, B on (S, h) , we define $A \cdot B = \text{tr}_h(A \otimes B)$, where the trace is taken with respect to the first and third indices).

Proposition 1 *Let S be a spacelike surface with embedding Φ_S . With the notation introduced above, the variation of the null expansion θ_ℓ along an arbitrary vector field ζ takes the form*

$$\delta_\zeta \theta_k = (\mathcal{L}_\zeta \mathbf{k}^{(\zeta)})(H) - \text{tr}_h(a^{\zeta, S} \cdot \chi_k) + h^{\alpha\beta} k^\gamma \left[\frac{1}{2} \nabla_\gamma a^\zeta_{\alpha\beta} - \nabla_\alpha a^\zeta_{\gamma\beta} \right] \Big|_S, \quad (3)$$

where $\mathbf{k}^{(\zeta)}$ is the one-form associated to $k^{(\zeta)}$, $a^{\zeta, S} \stackrel{\text{def}}{=} \Phi_S^*(a^\zeta)$ and h_β^α is the projector tangent to S (i.e. $h(v) = v^\parallel$).

In this paper we will use this result in the following slightly modified form:

Lemma 1 *With the same notation as in Proposition 1, we have*

$$\delta_\zeta \theta_k = (\mathcal{L}_\zeta \mathbf{k}^{(\zeta)})(H) - \text{tr}_h (a^{\zeta, S} \cdot \chi_k) - h^{\alpha\beta} k^\gamma (\nabla_\alpha \nabla_\beta \zeta_\gamma - R^\mu_{\alpha\beta\gamma} \zeta_\mu) \Big|_S.$$

Proof. The symmetry of $h^{\alpha\beta}$ allows us to write

$$h^{\alpha\beta} k^\gamma \left[\frac{1}{2} \nabla_\gamma a_{\alpha\beta}^\zeta - \nabla_\alpha a_{\gamma\beta}^\zeta \right] \Big|_S = \frac{1}{2} h^{\alpha\beta} k^\gamma \left[\nabla_\gamma a_{\alpha\beta}^\zeta - \nabla_\alpha a_{\gamma\beta}^\zeta - \nabla_\beta a_{\gamma\alpha}^\zeta \right] \Big|_S \quad (4)$$

Inserting $a_{\alpha\beta}^\zeta = \nabla_\alpha \zeta_\beta + \nabla_\beta \zeta_\alpha$ and using the Ricci and first Bianchi identities it follows

$$\nabla_\gamma a_{\alpha\beta}^\zeta - \nabla_\alpha a_{\gamma\beta}^\zeta - \nabla_\beta a_{\gamma\alpha}^\zeta = -2 (\nabla_\alpha \nabla_\beta \zeta_\gamma - R^\mu_{\alpha\beta\gamma} \zeta_\mu). \quad (5)$$

Combining (4) and (5) into (3) proves the Lemma. \square

The variation formulae above are specially relevant for marginally outer trapped surfaces (MOTS), which are spacelike surfaces such that its mean curvature H is everywhere parallel to one of the two null normals. Assuming that $H \propto \ell$, a MOTS is defined by the property that $\theta_\ell = 0$. An important notion related to MOTS is the so-called stability operator which is directly related to the first order variations of θ_ℓ described above. More specifically, consider section of the normal bundle which is nowhere tangent to ℓ . This section can be uniquely defined by a vector field $v \in \mathfrak{X}(S)^\perp$ of the form

$$v = -\frac{1}{2}k + V\ell.$$

where $V \in C^\infty(S, \mathbb{R})$. It is useful to define the Hodge dual (see e.g. [12])

$$v^\star \stackrel{\text{def}}{=} \frac{1}{2}k + V\ell, \quad (6)$$

which satisfies $\langle v^\star, v \rangle = 0$ and $\langle v^\star, v^\star \rangle = -\langle v, v \rangle$. For any smooth function ψ on S , the **stability operator** [3] L_v is the differential operator defined by $L_v \psi \stackrel{\text{def}}{=} \delta_{\psi v} \theta_\ell$. Expression (1) gives the explicit expression

$$L_v \psi = -\Delta_h \psi + 2\mathbf{s}_\ell(\nabla_h \psi) + \left(\text{div}_h \mathbf{s} - \|\mathbf{s}\|_h^2 + \frac{\text{Scal}_h}{2} - \text{Ein}(\ell, v^\star) - V \|\chi_\ell\|_h^2 \right) \psi. \quad (7)$$

We will denote by L_- the stability operator along $-\frac{1}{2}k$. Expression (6) implies

$$L_v = L_- - V(\text{Ein}(\ell, \ell) + \|\chi_\ell\|_h^2). \quad (8)$$

The stability operator, like any other second order elliptic operator on a compact manifold, admits a principal eigenvalue, which is the eigenvalue with lowest real part. This eigenvalue turns out to be real and have a one-dimensional eigenspace (see the discussion in [3]). We will denote the principal eigenvalue of L_v by λ_v . The eigenspace of λ_v is of the form $c\phi_v$ where $\phi_v > 0$ and $c \in \mathbb{R}$. According to the sign of the principal eigenvalue, a MOTS is called strictly stable along v if $\lambda_v > 0$, marginally stable along v if $\lambda_v = 0$ and unstable along v if $\lambda_v < 0$ [3]. A MOTS is stable along v iff $\lambda_v \geq 0$.

In the next section we consider totally geodesic null hypersurfaces, which by construction are foliated by MOTS, and we study the stability operator for those MOTS.

3 Totally geodesic null hypersurfaces

In this paper, a null hypersurface \mathcal{H} is a codimension-one embedded submanifold of M with degenerate first fundamental form. As before, we usually identify \mathcal{H} and its image in M . Let ℓ be a future-directed null normal to \mathcal{H} , namely a nowhere-zero vector field on \mathcal{H} which is null, future-directed and tangent to \mathcal{H} . This vector field is defined up to a rescaling $\ell' = F\ell$ where $F : \mathcal{H} \rightarrow \mathbb{R}^+$. The one form ℓ obtained by lowering its index defines a normal one-form to \mathcal{H} . It is a standard property that the integral curves of ℓ are geodesic, which implies the existence of smooth function (the so-called **surface gravity**) $\kappa_\ell : \mathcal{H} \rightarrow \mathbb{R}$ satisfying $\nabla_\ell \ell \stackrel{\mathfrak{N}}{=} \kappa_\ell \ell$. Under a rescaling $\ell' = F\ell$, κ_ℓ transforms as $\kappa_{\ell'} = F\kappa_\ell + \ell(F)$.

The second fundamental form of \mathcal{H} with respect to ℓ is defined as $K_\ell(X, Y) \stackrel{\text{def}}{=} \langle Y, \nabla_X \ell \rangle$, where X, Y are vector fields tangent to \mathcal{H} . Under a rescaling of ℓ we obviously have $K_{F\ell} = FK_\ell$. Null hypersurfaces have the property that any spacelike surface embedded in \mathcal{H} (i.e. such that the embedding $\Phi_S : S \rightarrow M$ satisfies $\Phi_S(S) \subset \mathcal{H}$) has the property that $\ell|_S$ is automatically a null normal to S . Moreover, given any pair $\{X, Y\}$ of vector fields tangent to S , the second fundamental form along $\ell|_S$ of S satisfies, at any $p \in S$, $\chi_{\ell|_S}(X, Y)|_p = K_\ell(X, Y)|_p$.

In this paper we are interested in **totally geodesic null hypersurfaces**, namely null hypersurfaces with identically vanishing second fundamental form K_ℓ (the definition is obviously independent of the choice of null normal). We assume further a topological condition, namely,

Topological condition: The topology of \mathcal{H} is $S \times \mathbb{R}$ where S is a closed $(n - 1)$ -dimensional manifold. Furthermore, the null normal ℓ is tangent to the \mathbb{R} factor.

(\star)

This topological condition implies \mathcal{H} is a trivial bundle (\mathcal{H}, S, π) where π is projection of $S \times \mathbb{R}$ into the first factor. Moreover, it also implies that spacelike surfaces embedded in \mathcal{H} are automatically sections of (\mathcal{H}, S, π) . An immediate consequence of the vanishing of the second fundamental form K_ℓ is that any embedded surface S in \mathcal{H} has vanishing second fundamental form χ_ℓ along the null normal $\ell|_S$. In particular, the null expansion θ_ℓ vanishes identically, so that S is a MOTS. Another standard consequence is that $\text{Ein}(\ell, \ell)|_{\mathcal{H}} = 0$, which follows immediately from the Raychaudhuri equation (or equivalently, from (1) with

$\psi = 0$).

Our aim in this section is to analyze the stability operator for sections of a totally geodesic null hypersurface \mathcal{H} satisfying the topological condition (\star) . Select a null normal ℓ and choose a section S_0 . We will denote by k the unique (once ℓ is selected) future-directed null vector orthogonal to S_0 and satisfying $\langle \ell, k \rangle|_{S_0} = -2$. Since totally geodesic null hypersurfaces satisfy $\text{Ein}(\ell, \ell) \stackrel{\text{?}}{=} 0$ and $\chi_\ell \stackrel{\text{?}}{=} 0$, it follows that the stability operator (8) of S_0 is independent of the section v of the normal bundle. Thus, the stability operator and the corresponding principal eigenvalue are properties of S_0 alone. We will denote them by L_{S_0} and λ_{S_0} respectively.

In order to find an expression for L_{S_0} which depends solely on geometric properties of \mathcal{H} and of S_0 , it is convenient to introduce the following tensor:

Definition 1 *Let \mathcal{H} be a totally geodesic null hypersurface and let ℓ be a null normal of \mathcal{H} . Extend ℓ arbitrarily to a neighbourhood of \mathcal{H} . The **non-isolation tensor** is the tensor \mathcal{N}^ℓ defined at $p \in \mathcal{H}$ by*

$$\begin{aligned} \mathcal{N}^\ell|_p : T_p\mathcal{H} \times T_p\mathcal{H} &\longrightarrow T_p\mathcal{H} \\ X_p, Y_p &\longrightarrow \mathcal{N}^\ell|_p(X_p, Y_p)^\gamma \stackrel{\text{def}}{=} X_p^\alpha Y_p^\beta (\nabla_\alpha \nabla_\beta \ell^\gamma - R_{\mu\alpha\beta}{}^\gamma \ell^\mu) |_p \end{aligned} \quad (9)$$

In order for this definition to make sense it is necessary to show that \mathcal{N}^ℓ is independent of the extension of ℓ and that the right-hand side of (9) is tangent to \mathcal{H} .

Lemma 2 *The tensor $\mathcal{N}^\ell|_p$ is well-defined.*

Proof. The fact that the second fundamental form K^ℓ vanishes implies that, for any pair of vector fields X, Y on \mathcal{H} , the vector field $\nabla_X Y$ is tangent to \mathcal{H} . This defines a connection on \mathcal{H} by $\mathcal{D}_X Y \stackrel{\text{def}}{=} \nabla_X Y$. Let us first show that $\mathcal{N}^\ell|_p(X, Y)$ is independent of the extension of ℓ . Extend arbitrarily $X|_p, Y|_p$ to tensor fields tangent to \mathcal{H} . We have

$$\begin{aligned} \mathcal{N}^\ell|_p(X_p, Y_p) &= (\nabla_X \nabla_Y \ell - \nabla_{\nabla_X Y} \ell - \text{Riem}(X, \ell)Y) |_p \\ &= (\mathcal{D}_X \mathcal{D}_Y \ell - \mathcal{D}_{\mathcal{D}_X Y} \ell - \text{Riem}(X, \ell)Y) |_p, \end{aligned}$$

which only depends on the values of $\ell|_{\mathcal{H}}$. In order to show that $\mathcal{N}^\ell|_p(X_p, Y_p)$ is tangent to \mathcal{H} we calculate the commutator of \mathcal{L}_ℓ and D . Let X, Y, Z be arbitrary spacetime vector fields. The following identity is well-known (and easy to prove)

$$(\mathcal{L}_Z (\nabla_X Y) - \nabla_{[Z, X]} Y - \nabla_X (\mathcal{L}_Z Y))^\gamma = X^\alpha Y^\beta (\nabla_\alpha \nabla_\beta Z^\gamma - R_{\mu\alpha\beta}{}^\gamma Z^\mu)$$

Thus, for X, Y tangent to \mathcal{H} ,

$$\begin{aligned} \mathcal{L}_\ell (\mathcal{D}_X Y) - \mathcal{D}_{[\ell, X]} Y - \mathcal{D}_X (\mathcal{L}_\ell Y) &= \mathcal{L}_\ell (\nabla_X Y) - \mathcal{D}_{[\ell, X]} Y - \mathcal{D}_X (\mathcal{L}_\ell Y) \\ &= \nabla_{[\ell, X]} Y + \nabla_X (\mathcal{L}_\ell Y) + \mathcal{N}^\ell(X, Y) - \mathcal{D}_{[\ell, X]} Y - \mathcal{D}_X (\mathcal{L}_\ell Y) \\ &= \mathcal{N}^\ell(X, Y). \end{aligned} \quad (10)$$

Since the left-hand side is tangent to \mathcal{H} , so is the right-hand side. \square

Remark. The name **non-isolation tensor** comes from the concept of isolated horizon. As mentioned in the Introduction the definition of isolated horizon requires that the Lie derivative along ℓ and the covariant derivative \mathcal{D} commute. Let, as before, X, Y be arbitrary vector fields tangent to \mathcal{H} . The tensor $[\mathcal{L}_\ell, \mathcal{D}]Y$ is a one-covariant, one-contravariant tensor which acts on X according to

$$([\mathcal{L}_\ell, \mathcal{D}]Y)(X) = (\mathcal{L}_\ell(\mathcal{D}Y))(X) - \mathcal{D}_X(\mathcal{L}_\ell Y) = \mathcal{L}_\ell(\mathcal{D}_X Y) - \mathcal{D}_{[\ell, X]}Y - \mathcal{D}_X(\mathcal{L}_\ell Y) = \mathcal{N}^\ell(X, Y),$$

where in the second equality we have used the Leibniz rule $(\mathcal{L}_\ell T)(X) = \mathcal{L}_\ell(T(X)) - T(\mathcal{L}_\ell X)$ and the last equality follows from (10). Thus, we conclude that \mathcal{L}_ℓ and \mathcal{D} commute if and only if \mathcal{N}^ℓ vanishes. This remark also shows that the non-isolation tensor is a spacetime formulation of the tensor $-C_{ab}^c$ introduced in [7] (see eq. (4.3) there)

The following Proposition gives an explicit expression for the stability operator of S_0 in terms of the properties of S_0 and of the totally geodesic horizon. This result extends a previous result by Booth & Fairhurst valid for isolated horizons [11].

Proposition 2 (Stability operator of a section of a totally geodesic null horizon)

Let \mathcal{H} be a totally geodesic null hypersurface satisfying the topological condition (). Let ℓ be a null normal and S_0 a section of \mathcal{H} . Denote by k the unique future-directed null vector orthogonal to S_0 and satisfying $\langle \ell, k \rangle|_{S_0} = -2$. Then, the stability operator of S_0 reads*

$$L_{S_0}\psi = -\Delta_h\psi + 2\mathbf{s}_\ell(\nabla^h\psi) + \psi\left(2\text{div}_h\mathbf{s}_\ell - \frac{1}{2}\kappa_\ell\theta_k + \frac{1}{2}\langle k, \text{tr}_h\mathcal{N}^\ell \rangle\right) \quad (11)$$

Proof. The fact that \mathcal{H} is totally geodesic means precisely that the deformation tensor of ℓ vanishes when acting on tangent vectors to \mathcal{H} . This implies, in particular, that $a^{\ell, S_0} \stackrel{\text{def}}{=} \Phi_{S_0}(a^\ell) = 0$. Let $k^{(\ell)}$ be the vector field on \mathcal{H} defined by the property that $k^{(\ell)}$ is null, satisfies $\langle \ell, k^{(\ell)} \rangle \stackrel{\text{def}}{=} -2$ and is orthogonal to $\varphi_\tau(S_0)$, where φ_τ is the local one-parameter group of transformations generated by ℓ . Under these conditions we can apply Lemma 1 with $\zeta = \ell$. Since the mean curvature of S_0 is $H = -\frac{1}{2}\theta_k\ell$, it follows

$$(\mathcal{L}_\ell \mathbf{k}^{(\ell)})(H) = -\frac{1}{2}\theta_k(\mathcal{L}_\ell \mathbf{k}^{(\ell)})(\ell) = -\frac{1}{2}\theta_k\mathcal{L}_\ell(\langle \ell, k^{(\ell)} \rangle) = 0.$$

Using the definition of \mathcal{N}^ℓ we conclude, from Lemma 1,

$$\delta_\ell\theta_k = -\langle k, \text{tr}_h\mathcal{N}^\ell \rangle|_{S_0}. \quad (12)$$

Substituting $\alpha = 0$ and $\psi = -2$ in (2) yields

$$\delta_\ell\theta_k = 2\text{div}_h\mathbf{s}_\ell + 2\|\mathbf{s}_\ell\|_h^2 - \text{Scal}_h + \text{Ein}(\ell, k) - \kappa_\ell\theta_k. \quad (13)$$

Combining (12) and (13) it follows

$$\text{Ein}(\ell, k) \stackrel{S_0}{=} \text{Scal}_h - 2\text{div}_h\mathbf{s}_\ell - 2\|\mathbf{s}_\ell\|_h^2 + \kappa_\ell\theta_k - \langle k, \text{tr}_h\mathcal{N}^\ell \rangle \quad (14)$$

Substituting this into (1) with $\alpha = 0$ we find

$$L_{S_0}(\psi) \stackrel{\text{def}}{=} \delta_{-\frac{\psi}{2}k} \theta_\ell = -\Delta_h \psi + 2\mathbf{s}_\ell(\nabla^h \psi) + \psi \left(2\text{div}_h \mathbf{s}_\ell - \frac{1}{2} \kappa_\ell \theta_k + \frac{1}{2} \langle k, \text{tr}_h \mathcal{N}^\ell \rangle \right)$$

which proves the Proposition. \square

We note for later use the following expression obtained in (14) (c.f. expression (8.16) in ??)

Corollary 1 *The Einstein tensor of $(\mathcal{M}, g^{(n+1)})$ satisfies*

$$\text{Ein}(\ell, k) \stackrel{S_0}{=} \text{Scal}_h - 2\text{div}_h \mathbf{s}_\ell - 2\|\mathbf{s}_\ell\|_h^2 + \kappa_\ell \theta_k - \langle k, \text{tr}_h \mathcal{N}^\ell \rangle. \quad (15)$$

Recall that on a compact Riemannian manifold (S_0, h) , any one-form ω admits a **Hodge decomposition**, i.e. can be written as the sum of a gradient and a divergence-free one-form, namely $\omega = df + \sigma$ where $f \in C^\infty(S, \mathbb{R})$ and σ satisfies $\text{div}_h \sigma = 0$. This decomposition is unique except for an additive constant in f .

Corollary 2 *With the same notation and assumptions as in Proposition 2. Let $\mathbf{s}_\ell = dQ_\ell + \mathbf{z}$ be the Hodge decomposition of \mathbf{s}_ℓ and define $u_\ell \stackrel{\text{def}}{=} e^{2Q_\ell}$. Then, the stability operator of S_0 is*

$$L_{S_0}(\psi) = -\text{div}_h \left(u_\ell \nabla_h \left(\frac{\psi}{u_\ell} \right) \right) + 2\mathbf{z}(\nabla_h \psi) + \frac{1}{2} (\langle k, \text{tr}_h \mathcal{N}^\ell \rangle - \kappa_\ell \theta_k) \psi, \quad (16)$$

Proof. Straightforward calculation. \square

Remark. Note that we have not added a subindex ℓ to the one-form \mathbf{z} . This is because this tensor is independent of the choice of ℓ . Indeed, consider any other null normal $\ell' = F\ell$, with $F \in C^\infty(\mathcal{H}, \mathbb{R}^+)$ and define $F_0 \stackrel{\text{def}}{=} F|_{S_0}$. From the definition of connection one-form we have $\mathbf{s}_{\ell'} = \mathbf{s}_\ell + d \ln F_0$. Consequently \mathbf{z} does not depend on ℓ and $Q_{\ell'} = Q_\ell + \ln F_0$ so that $u_{\ell'} = u_\ell F_0^2$.

This expression for the stability operator of MOTS is the key to obtain the stability results below. It is clear that knowledge of the non-isolation tensor is necessary in order to draw any conclusions. It is reasonable to start with the simplest possible case, namely when this tensor vanishes identically. This case is relevant because it includes all isolated horizons and all Killing horizons (provided the topological condition \star is satisfied).

4 Stability of MOTS in non-evolving horizons

We start with the following definition, which extends the usual notion of isolated horizon and has the advantage that it includes all Killing horizons, irrespectively of whether the spacetime satisfies suitable energy conditions or not.

Definition 2 Let $(M, g^{(n+1)})$ be a spacetime. A **non-evolving horizon** (\mathcal{H}, ℓ) is a totally geodesic null hypersurface of $(M, g^{(n+1)})$ endowed with a future-directed null normal ℓ such that the non-isolation tensor $\mathcal{N}^{(\ell)}$ vanishes identically.

This definition includes all Killing horizons because Killing vectors satisfy the well-known identity

$$\nabla_\alpha \nabla_\beta \xi_\gamma = \xi_\mu R^\mu_{\alpha\beta\gamma}.$$

Thus, with the canonical choice $\ell = \xi|_{\mathcal{H}_\xi}$ the hypersurface \mathcal{H}_ξ satisfies $\mathcal{N}^\ell = 0$ and $(\mathcal{H}_\xi, \xi|_{\mathcal{H}_\xi})$ is a non-evolving horizon.

Recall that a spacelike surface is future marginally trapped if its mean curvature vector H is future-directed and tangent everywhere to one of its null normals. The following result is an easy consequence of Corollary 2.

Proposition 3 Let (\mathcal{H}, ℓ) be a non-evolving horizon satisfying the topological condition (\star) and let S_0 be any section of \mathcal{H} . Assume that S_0 is future marginally trapped with non-identically vanishing mean curvature H . Then

- If the surface gravity κ_ℓ is positive on S_0 then S_0 is strictly stable.
- If the surface gravity κ_ℓ vanishes on S_0 then S_0 is marginally stable.
- If the surface gravity κ_ℓ is negative on S_0 then S_0 is unstable.

Remark. As already mentioned, isolated horizons always have constant surface gravity κ_ℓ . Thus, Proposition 3 provides a stability classification for future marginally trapped, non-minimal, sections of isolated horizons. The sign of κ_ℓ determines the stability character of S_0 . In the particular case of axially symmetric isolated horizons with topology $\mathbb{S}^2 \times \mathbb{R}$ and assuming that the mean curvature of S_0 is nowhere zero, this result has been obtained by Booth and Fairhurst [11]. A generalization with the same symmetry and topology assumptions but assuming only $\theta_k \leq 0$ and negative somewhere has been obtained by Jaramillo [31].

Proof. Let λ_{S_0} be the principal eigenvalue of the stability operator L_{S_0} . Let ϕ_0 be principal eigenfunction $L_{S_0}(\phi_0) = \lambda_{S_0}\phi_0$. Integrating this equation on S_0 we get, from (16) with $\mathcal{N}^\ell = 0$,

$$\lambda_{S_0} \int_{S_0} \phi_0 \boldsymbol{\eta}_{S_0} = -\frac{1}{2} \int_{S_0} \kappa_\ell \theta_k \phi_0 \boldsymbol{\eta}_{S_0}, \quad (17)$$

where $\boldsymbol{\eta}_{S_0}$ is the volume form of (S, h) and we have used the fact that $\mathbf{z}(\nabla^h \psi)$ is a divergence for any ψ and hence integrates to zero. Since ϕ_0 has constant sign, the claims of the Proposition follow directly from (17) after using the fact that under the conditions of the Proposition $\theta_k|_{S_0} \leq 0$ and not identically zero. \square

In fact, in the degenerate case (i.e. when the surface gravity vanishes) the argument proves a stronger statement.

Corollary 3 *Let (\mathcal{H}, ℓ) be a non-evolving horizon and let S_0 be any section of \mathcal{H} . If κ_ℓ vanishes on S_0 then S_0 is marginally stable.*

Having derived a general expression of the stability operator of sections of a totally geodesic null hypersurface, it is natural to ask how does this operator depend on the section. In order to determine the dependence on the section, we need to compare two arbitrary sections on \mathcal{H} . Let us therefore fix a section S_0 on \mathcal{H} and define a function τ in \mathcal{H} by $\ell(\tau) = 1$ and $\tau|_{S_0} = 0$. τ is a coordinate along the \mathbb{R} factor in \mathcal{H} . Any other section $S[f]$ can then be defined as a graph over S_0 , $f : S_0 \rightarrow \mathbb{R}$. Let $\pi_f : S[f] \rightarrow S_0$ be the natural projection along orbits of ℓ . π_f is a diffeomorphism and in fact an isometry between these two spaces with their respective induced metrics. Since the derivation holds independently of whether \mathcal{N}^ℓ vanishes or not, we state the result for general totally geodesics null hypersurfaces.

In order to determine the behaviour of the stability operator, we need to relate the connection one-forms \mathbf{s}_ℓ of S_0 and $\mathbf{s}_\ell[\mathbf{f}]$ of $S[f]$ and the null expansions θ_k of S_0 and $\theta_k[f]$ of $S[f]$. A related result written in terms of the induced connection of the null hypersurface was obtained in [7] (see also [20])

Lemma 3 *Let \mathcal{H} be a totally geodesic null hypersurface satisfying the topological condition (\star) and let ℓ a null normal to \mathcal{H} . Fix a section S_0 and define $S[f]$ and π_f as before. Let k_f be the null normal to $S[f]$ satisfying $\langle \ell, k_f \rangle \stackrel{S[f]}{=} -2$. Denote by $\mathbf{s}_\ell[\mathbf{f}]$ the normal connection of $S[f]$, by $\chi^k[f]$ the second fundamental form of $S[f]$ along k_f and by $\theta_k[f]$ its trace. Then*

$$\mathbf{s}_\ell[\mathbf{f}] = \pi_f^*(\mathbf{s}_\ell + \kappa_\ell df), \quad (18)$$

$$\chi^k[f] = \pi_f^*(\chi^k + 2\text{Hess}_h f + 2\kappa_\ell df \otimes df + 2df \otimes \mathbf{s}_\ell + 2\mathbf{s}_\ell \otimes df) \quad (19)$$

$$\theta_k[f] = (\theta_k + 2\Delta_h f + 2\kappa_\ell \|\nabla^h f\|_h^2 + 4\mathbf{s}_\ell(\nabla^h f)) \circ \pi_f \quad (20)$$

where \mathbf{s}_ℓ , χ^k and θ_k are, respectively, $\mathbf{s}_\ell[f=0]$, $\chi^k[f=0]$ and $\theta_k[f=0]$ and $\text{Hess}_h f$ is the Hessian of f with the metric h .

Proof. As before, let φ_τ be the local one-parameter group of diffeomorphisms of \mathcal{H} generated by ℓ and $S_\tau = \varphi_\tau(S_0)$. Let k be the unique vector field on \mathcal{H} with the properties of being null, orthogonal to S_τ and satisfying $\langle k, \ell \rangle = -2$. Given any vector field X defined on S_0 and tangent to \mathcal{H} , we consistently extend it to \mathcal{H} by solving $\mathcal{L}_\ell X = 0$ with this initial data. This vector field is obviously tangent to S_τ . We will refer to any such vector field as a *vector field tangent to S_0* although it is defined everywhere on \mathcal{H} . Any scalar function $w \in C^\infty(S_0, \mathbb{R})$ is also extended to \mathcal{H} by solving $\mathcal{L}_\ell w = 0$ with this initial data.

With this notation it is immediate to check that $X_f \stackrel{\text{def}}{=} (\pi_f^{-1})_*(X) = X + X(f)\ell$. For the null vector k_f we note that $\langle k_f, \ell \rangle \stackrel{S[f]}{=} -2$ implies $k_f \stackrel{S[f]}{=} k + A\ell + Z$ for some function A and some vector field Z tangent to S_0 . Multiplying by X_f we get

$$0 = \langle k_f, X_f \rangle = -2X(f) + \langle Z, X \rangle_h,$$

which implies $Z \stackrel{\mathfrak{H}}{=} 2\nabla^h f$. The condition $\langle k_f, k_f \rangle \stackrel{S[f]}{=} 0$ fixes $A \stackrel{\mathfrak{H}}{=} \|\nabla^h f\|_h^2$. Thus,

$$k_f \stackrel{S[f]}{=} k + \|\nabla^h f\|_h^2 \ell + 2\nabla^h f. \quad (21)$$

Let us start with the calculation of $\mathbf{s}_\ell[\mathbf{f}]$. From its definition,

$$\begin{aligned}
\mathbf{s}_\ell[\mathbf{f}](X_f) &\stackrel{\text{def}}{=} -\frac{1}{2}\langle k_f, \nabla_{X_f}\ell \rangle \Big|_{S[f]} \\
&= -\frac{1}{2}\langle k + \|\nabla^h f\|_h^2 \ell + 2\nabla^h f, \nabla_X \ell + X(f)\kappa_\ell \ell \rangle \\
&= \mathbf{s}_\ell(X) + \kappa_\ell X(f),
\end{aligned} \tag{22}$$

where in the last equality we have used the fact that ℓ is orthogonal to any vector tangent to \mathcal{H} and

$$\langle \nabla^h f, \nabla_X \ell \rangle = 0, \tag{23}$$

which is a consequence of the vanishing of second fundamental form K^ℓ of \mathcal{H} . Now, since $\pi_f^*(\mathbf{s}_\ell)(X_f) = \mathbf{s}_\ell[(\pi_f)_*(X_f)] = \mathbf{s}_\ell[(\pi_f)_* \circ (\pi_f^{-1})_*(X)] = \mathbf{s}_\ell(X)$, (22) can be written as

$$\mathbf{s}_\ell[\mathbf{f}](X_f) = \pi_f^*(\mathbf{s}_\ell + \kappa_\ell df)(X_f)$$

which proves (18). In order to show (19) we need to evaluate the second fundamental form of $S[f]$ along k_f . Let X, Y be a pair of vector fields tangent to S_0 and X_f, Y_f the corresponding fields tangent to $S[f]$. Using $\mathcal{L}_\ell Y = 0$ (in the form $\nabla_\ell Y = \nabla_Y \ell$), it follows by straightforward calculation

$$\nabla_{X_f} Y_f \stackrel{S[f]}{=} \nabla_X Y + [X(Y(f)) + \kappa_\ell X(f)Y(f)]\ell + X(f)\nabla_Y \ell + Y(f)\nabla_X \ell.$$

Multiplying this expression with k_f , as given in (21), and using (23) we find

$$\begin{aligned}
\chi^k[f](X_f, Y_f) &\stackrel{\text{def}}{=} -\langle k_f, \nabla_{X_f} Y_f \rangle \Big|_{S[f]} \\
&= \chi^k(X, Y) + 2[X(Y(f)) - \langle \nabla^h f, \nabla_X Y \rangle + \kappa_\ell X(f)Y(f)] \\
&\quad + 2[X(f)\mathbf{s}_\ell(Y) + Y(f)\mathbf{s}_\ell(X)].
\end{aligned} \tag{24}$$

Now, by definition of Hessian, $\text{Hess}_h f(X, Y) = X(Y(f)) - \langle \nabla^h f, \nabla_X Y \rangle$, and (24) becomes

$$\begin{aligned}
\chi^k[f](X_f, Y_f) &= \chi^k(X, Y) + 2\text{Hess}_h f(X, Y) + 2\kappa_\ell X(f)Y(f) + 2[X(f)\mathbf{s}_\ell(Y) + Y(f)\mathbf{s}_\ell(X)] \\
&= \pi_f^*(\chi^k + 2\text{Hess}_h f + 2\kappa_\ell df \otimes df + 2df \otimes \mathbf{s}_\ell + 2\mathbf{s}_\ell \otimes df)(X_f, Y_f).
\end{aligned}$$

This establishes (19). Taking the trace with the metric of $S[f]$ and using that π_f is an isometry gives (20). \square

With this transformation lemma at hand, we can relate the stability operators of S_0 and of $S[f]$ in the particular case of constant surface gravity. As discussed above, this is the physically most interesting situation since it holds for any spacetime satisfying the dominant energy condition.

Proposition 4 (Dependence of the stability operator on the section) *Let (\mathcal{H}, ℓ) be a non-evolving horizon satisfying the topological condition (\star) and assume that the surface gravity κ_ℓ is constant. Then the stability operator of $S[f]$ is related to the stability operator of S_0 by*

$$L_{S[f]}(\psi \circ \pi_f) = e^{\kappa_\ell f} L_{S_0}(e^{-\kappa_\ell f} \psi) \circ \pi_f, \quad \forall \psi \in C^2(S_0, \mathbb{R}).$$

Proof. The equality is tensorial, so it suffices to work in a suitable coordinate system. Select any point $p \in S_0$ and a coordinate system $\{U_p, y^A\}$ near p ($A, B = 1, \dots, n-1$). Then $\{\pi_f^{-1}(U_p), y^A \circ \pi_f\}$ is a coordinate system near $\pi_f^{-1}(p)$. In these coordinates, π_f is the identity map and we can simply write

$$\begin{aligned} s_\ell[f]_A &= s_{\ell A} + \kappa_\ell \nabla_A^h f, \\ \theta_k[f] &= \theta_k + 2\Delta_h f + 2\kappa_\xi \nabla_A^h f \nabla^h A f + 4s_\ell^A \nabla_A^h f. \end{aligned}$$

Applying (11) to the function $w\psi$ and expanding derivatives of products it follows, assuming $w > 0$ everywhere,

$$\frac{1}{w} L_{S_0}(w\psi) = -\Delta_h \psi + 2 \left(s_\ell^A - \frac{\nabla^h A w}{w} \right) \nabla_A^h \psi + \left(2\nabla_A^h s_\ell^A - \frac{1}{2} \kappa_\ell \theta_k + 2s_\ell^A \frac{\nabla_A^h w}{w} - \frac{1}{w} \Delta_h w \right) \psi. \quad (25)$$

On the other hand, Lemma 3 and (11) imply that the stability operator for $S[f]$ reads, in these coordinates (recall that π_f is an isometry)

$$\begin{aligned} L_{S[f]}(\psi) &= -\Delta_h \psi + 2s_\ell[f]^A \nabla_A^h \psi + \left(2\nabla_A^h s_\ell[f]^A - \frac{1}{2} \kappa_\ell \theta_k[f] \right) \psi \\ &= -\Delta_h \psi + 2 \left(s_\ell^A + \kappa_\ell \nabla^h A f \right) \nabla_A^h \psi + \left(2\nabla_A^h s_\ell^A + \kappa_\ell \Delta_h f - \right. \\ &\quad \left. \frac{1}{2} \kappa_\ell (\theta_k + 2\kappa_\ell \nabla_A^h f \nabla^h A f + 4s_\ell^A \nabla_A^h f) \right) \psi, \end{aligned} \quad (26)$$

where we have used that κ_ℓ is constant. We want to find w such that the right hand sides of (25) and of (26) are the same. The term in $\nabla_A^h \psi$ forces $w = e^{-\kappa_\ell f}$ up to an irrelevant multiplicative constant. Inserting this expression in the zero order term of (25) we get the zero order term of (26), which proves the claim. \square

Corollary 4 *Let (\mathcal{H}, ℓ) be a non-evolving horizon satisfying the topological condition (\star) with constant surface gravity. Then the principal eigenvalue is independent of the section.*

Proof. Let ϕ_0 be the principal eigenfunction and λ_{S_0} the principal eigenvalue of S_0 . Define $\phi_f \equiv (e^{\kappa_\ell f} \phi_0) \circ \pi_f$. Then

$$L_{S[f]}(\phi_f) = L_{S[f]}((e^{\kappa_\ell f} \phi_0) \circ \pi_f) = (e^{\kappa_\ell f} L_{S_0}(\phi_0)) \circ \pi_f = \lambda_{S_0}(e^{\kappa_\ell f} \phi_0) \circ \pi_f = \lambda_{S_0} \phi_f$$

Thus, ϕ_f is an eigenfunction of $L_{S[f]}$ which must be the principal eigenfunction because it does not change sign. \square

This corollary states that, whenever the surface gravity is constant, the stability eigenvalue is a property of the non-evolving horizon (\mathcal{H}, ℓ) instead of a property of any particular section. In the constant surface gravity case, we will denote this eigenvalue by $\lambda_{\mathcal{H}}$ and we will say that the horizon is stable, marginally stable or unstable depending on the sign of $\lambda_{\mathcal{H}}$.

Proposition 4 and Corollary 4 give a fully satisfactory answer to the issue of how does stability depend on the section in the constant surface gravity case. It is quite natural to ask whether the independence of the eigenvalue on the section also holds in the case of non-constant surface gravity. The following lemma shows that Corollary 4 is not true when κ_{ℓ} is not a constant (which, recall, can only happen if the dominant energy condition does not hold).

Lemma 4 *There exist Killing horizons with non-constant surface gravity for which the principal eigenvalue depends on the section.*

Proof. It suffices to give an explicit example. Consider the four-dimensional spacetime $\mathbb{R}^2 \times \mathbb{S}^2$ with metric

$$ds^2 = -2xWdt^2 + 2dtdx + \gamma,$$

where γ is the standard metric on the sphere and $W : \mathbb{S}^2 \rightarrow \mathbb{R}$ is a smooth function which we take to be constant along one of the Killing vectors of (\mathbb{S}^2, γ) . It is clear that $\xi = \partial_t$ is a Killing vector and $\mathcal{H}_{\xi} \stackrel{\text{def}}{=} \{x = 0\}$ is a Killing horizon of topology $\mathbb{R} \times \mathbb{S}^2$. An immediate calculation shows that the surface gravity of \mathcal{H}_{ξ} is $\kappa_{\xi} = W$ and that the connection one-form \mathbf{s}_{ξ} and the null expansion θ_k of the section $S_0 \stackrel{\text{def}}{=} \{t = 0\}$ vanish identically. Using the transformation Lemma 3 we can write down $s_{\xi}[f]$ and $\theta_k[f]$ for any surface $S[f]$ defined by a graph function $f : \mathbb{S}^2 \rightarrow \mathbb{R}$. Substitution into (11) (with $\mathcal{N}^{\ell} = 0$) gives the stability operator $L_{S[f]}$. Explicitly

$$L_{S[f]}\psi = -\Delta_{\gamma}\psi + 2W\langle\nabla^{\gamma}f, \nabla^{\gamma}\psi\rangle + (W\Delta_{\gamma}f + 2\langle\nabla^{\gamma}W, \nabla^{\gamma}f\rangle - W^2\|\nabla^{\gamma}f\|^2)\psi.$$

We first notice that the principal eigenvalue of the section $\{f = 0\}$ is $\lambda_{S_0} = 0$. Next, select f to be invariant under the same Killing vector of (\mathbb{S}^2, γ) as W . Then Wdf is a closed one-form in \mathbb{S}^2 . Define V as any solution of $dV = -2Wdf$ and let $U = e^V$. The stability operator becomes

$$L_{S[f]}\psi = -e^{-V}\text{div}_{\gamma}(e^V\nabla^{\gamma}\psi) + (W\Delta_{\gamma}f + 2\langle\nabla^{\gamma}W, \nabla^{\gamma}f\rangle - W^2\|\nabla^{\gamma}f\|^2)\psi$$

This is a self-adjoint operator with respect to the L^2 product with measure $\boldsymbol{\eta}_V \stackrel{\text{def}}{=} e^V\boldsymbol{\eta}_{\gamma}$. It follows that the principal eigenvalue is given by the Rayleigh-Ritz quotient

$$\lambda_{S[f]} = \inf_{\psi \neq 0} \frac{\int_{\mathbb{S}^2} \|\nabla^{\gamma}\psi\|^2 + (W\Delta_{\gamma}f + 2\langle\nabla^{\gamma}W, \nabla^{\gamma}f\rangle - W^2\|\nabla^{\gamma}f\|^2)\psi^2 \boldsymbol{\eta}_V}{\int_{\mathbb{S}^2} \psi^2 \boldsymbol{\eta}_V}. \quad (27)$$

To prove the lemma, we only need to find a pair of functions $\{W, f\}$ for which $\lambda_{S[f]} \neq 0$. Choose $f = a \cos \theta$ and $W = b \cos \theta$, where $\{\theta, \varphi\}$ are standard angular coordinates on the

sphere and $\{a, b\}$ are constants. Choose $\psi = 1$ in the Rayleigh-Ritz quotient (27). This gives an upper bound for $\lambda_{S[f]}$, namely

$$\lambda_{S[f]} \leq \frac{\int_0^\pi e^{c \cos^2 \theta} (2c (\cos^2 \theta - \sin^2 \theta) - c^2 \sin^2 \theta \cos^2 \theta) \sin \theta d\theta}{\int_0^\pi e^{c \cos^2 \theta} \sin \theta d\theta} \stackrel{\text{def}}{=} I(c)$$

where $c \stackrel{\text{def}}{=} -ab$. Expanding the integrands near $c = 0$, the following expression is obtained

$$I(c) = -\frac{2c}{3} + \frac{2c^2}{9} + o(c^3).$$

If c is positive and close to zero, then $I(c) < 0$ and the corresponding eigenvalue is also negative, which proves the claim. \square

We have shown before that degenerate non-evolving horizons are necessarily marginally stable. It is natural to ask whether non-degenerate horizons should be non-marginally stable. The following Proposition states that non-degenerate, marginally stable, non-evolving horizons can be characterized by the existence of minimal sections. More precisely,

Theorem 1 *Let (\mathcal{H}, ℓ) be a non-evolving horizon satisfying the topological condition (\star) with constant surface gravity $\kappa_\ell \neq 0$. If \mathcal{H} admits a minimal section (i.e. a section S_0 with vanishing mean curvature vector $H|_{S_0} = 0$), then \mathcal{H} is marginally stable.*

Conversely, if \mathcal{H} is marginally stable, then there exists a section $S[f]$ satisfying

$$\int_{S[f]} \theta_k[f] \boldsymbol{\eta}_{S[f]} = 0. \quad (28)$$

If, moreover, for some section S_0 , the Hodge decomposition $\mathbf{s}_\ell = \mathbf{z} + dQ_\ell$ satisfies (i) $\mathbf{z}(\nabla^h Q_\ell) = 0$ and (ii) $\mathbf{z}(\nabla^h \phi_0) = 0$, where ϕ_0 is the principal eigenfunction of L_{S_0} , then the section $S[f]$ is, in fact, minimal.

Proof. Assume that \mathcal{H} admits a section S_0 which vanishing mean curvature. In particular, its null expansion θ_k vanishes. The same proof as for Proposition 3 implies $\lambda_{S_0} = 0$, and the first claim is proved.

For the (partial) converse, assume that $\lambda_{\mathcal{H}} = 0$. Choose any section S_0 and decompose $\mathbf{s}_\ell = \frac{du_\ell}{2u_\ell} + \mathbf{z}$. According to Lemma 3, for any section $S[f]$ defined by a graph on S_0 the connection one-form is $\mathbf{s}_\ell[\mathbf{f}] = \pi_f^*(\mathbf{s}_\ell + \kappa_\ell df)$. Its Hodge decomposition is therefore $\mathbf{s}_\ell[\mathbf{f}] = \frac{du_\ell[\mathbf{f}]}{2u_\ell[\mathbf{f}]} + \mathbf{z}[\mathbf{f}]$, where

$$u_\ell[f] = (e^{2\kappa_\ell f} u_\ell) \circ \pi_f, \quad \mathbf{z}[\mathbf{f}] = \pi_f^*(\mathbf{z}) \quad (29)$$

(this shows in particular that \mathbf{z} is independent of the section in the constant surface gravity case). Let ϕ_0 a principal eigenfunction of L_{S_0} and define

$$f \stackrel{\text{def}}{=} \frac{1}{\kappa_\ell} \ln \left(\frac{\phi_0}{u_\ell} \right).$$

Proposition 4 gives

$$L_{S[f]}(u_\ell[f]) = e^{\kappa_\ell f} L_{S_0}(e^{\kappa_\ell f} u_\ell) \circ \pi_f = e^{\kappa_\ell f} L_{S_0}(\phi_0) \circ \pi_f = 0.$$

Now, the left-hand side can be evaluated using (16) applied to the section $S[f]$. It follows

$$\begin{aligned} \theta_k[f] &= \frac{4}{\kappa_\ell} \mathbf{z}[\mathbf{f}] (\nabla^h \ln(u_\ell[f])) \\ &= \frac{4}{\kappa_\ell} \mathbf{z} (\nabla^h \ln(e^{2\kappa_\ell f} u_\ell)) \circ \pi_f \\ &= \frac{4}{\kappa_\ell} \mathbf{z} \left(\nabla^h \ln \left(\frac{\phi_0^2}{u_\ell} \right) \right) \circ \pi_f. \end{aligned} \quad (30)$$

The integral of the right hand side on $S[f]$ is identically zero. This proves (28). The last claim is immediate from (30). \square

Remark. This Proposition implies that marginally stable, non-degenerate, non-evolving horizons can be locally foliated by sections with vanishing total null mean curvature. Whenever (i) and (ii) are satisfied this foliation is by minimal sections. This follows because given a section with these properties a local foliation is obtained by dragging along the null normal ℓ or, alternatively, by noticing that the proof applies also to the graph function $f = \kappa_\ell^{-1} \ln(\phi_0 u^{-1}) + a$ with a an arbitrary constant.

In expression (17) we have found an integral equality for θ_k on any section of a non-evolving horizon. This inequality implies, in particular, that on any stable section S_0 we have $\int_{S_0} \kappa_\ell \theta_k \phi_0 \boldsymbol{\eta}_{S_0} \leq 0$ and zero if and only if the section is marginally stable. This type of inequalities are useful but have the potential drawback that they involve the principal eigenfunction, which typically is not known explicitly. Our final aim in this section is to obtain an integral expression which involves computable functions on any section S_0 .

Let (\mathcal{H}, ℓ) be a non-evolving horizon and let S_0 any section. Let ϕ_0 the principal eigenvalue of L_{S_0} . Applying (16) to $\psi = \phi_0$ we get

$$-\frac{1}{2} \kappa_\ell \theta_k \phi_0 = \lambda_{S_0} \phi_0 + \operatorname{div}_h \left(u_\ell \nabla^h \left(\frac{\phi_0}{u_\ell} \right) \right) - 2\mathbf{z}(\nabla^h \phi_0).$$

In order to get rid of ϕ_0 we calculate

$$\begin{aligned} -\frac{1}{2} \kappa_\ell \theta_k u_\ell &= \frac{u_\ell}{\phi_0} \left(-\frac{1}{2} \kappa_\ell \theta_k \phi_0 \right) \\ &= \lambda_{S_0} u_\ell + \frac{u_\ell}{\phi_0} \left[\operatorname{div}_h \left(u_\ell \nabla^h \left(\frac{\phi_0}{u_\ell} \right) \right) - 2\mathbf{z}(\nabla^h \phi_0) \right] \\ &= \lambda_{S_0} u_\ell + \operatorname{div}_h \left(\frac{u_\ell^2}{\phi_0} \nabla^h \left(\frac{\phi_0}{u_\ell} \right) \right) + \frac{u_\ell^3}{\phi_0} \left\| \nabla^h \left(\frac{\phi_0}{u_\ell} \right) \right\|_h^2 - \frac{2u_\ell}{\phi_0} \mathbf{z}(\nabla^h \phi_0). \end{aligned} \quad (31)$$

This identity implies the following result.

Proposition 5 *Let (\mathcal{H}, ℓ) be a non-evolving horizon satisfying the topological condition (\star) and S_0 a section. Assume that the one-form \mathbf{z} in the Hodge decomposition $\mathbf{s}_\ell = \frac{du_\ell}{2u_\ell} + \mathbf{z}$ satisfies $\mathbf{z}(\nabla^h \phi_0) = 0$, where ϕ_0 is the principal eigenfunction of L_{S_0} . Then*

- (i) *If S_0 is strictly stable then $\int_{S_0} \kappa_\ell \theta_k u_\ell < 0$.*
- (ii) *If S_0 is marginally stable then $\int_{S_0} \kappa_\ell \theta_k u_\ell \leq 0$, and it vanishes if and only if u_ℓ is a principal eigenfunction of L_{S_0} .*
- (iii) *If S_0 is marginally stable and κ_ℓ is constant and non-zero on S_0 , then $\int_{S_0} \kappa_\ell \theta_k u_\ell \leq 0$, and zero if and only if $\theta_k = 0$.*

Proof. Integrating (31) and using $\mathbf{z}(\nabla^h \phi_0) = 0$ yields

$$-\int_{S_0} \kappa_\ell \theta_k u_\ell = 2\lambda_{S_0} \int_{S_0} u_\ell \boldsymbol{\eta}_{S_0} + \int_{S_0} \frac{2u_\ell^3}{\phi_0} \left\| \nabla^h \left(\frac{\phi_0}{u_\ell} \right) \right\|_h^2 \boldsymbol{\eta}_{S_0}.$$

If $\lambda_{S_0} > 0$ then the right-hand side is strictly positive, which proves claim (i). If $\lambda_{S_0} = 0$ the right-hand side is non-negative and zero if and only if $u_\ell = c\phi_0$ for some constant c . This proves claim (ii). For the third claim we only need to show that if the integral vanishes then $\theta_k = 0$. Since by point (ii), $u = c\phi_0$, substitution into (31) implies $\theta_k = 0$. \square

Propositions 1 and 5 require hypotheses involving the orthogonality between \mathbf{z}_ℓ and the gradient of various functions (the eigenfunction ϕ_0 and the Hodge dual function u_ℓ in Proposition (1) and ϕ_0 in Proposition (5)). The simplest case where these hypotheses are satisfied involve Killing horizons of static Killing vectors with sufficiently simple topology. More precisely

Lemma 5 *Let \mathcal{H}_ξ be a Killing horizon in a spacetime $(\mathcal{M}, g^{(n+1)})$. Assume that the Killing vector ξ is integrable and that the Killing horizon satisfies the topological condition (\star) with S simply connected. Then \mathbf{z} vanishes on any section of \mathcal{H}_ξ .*

Proof. Let S_0 be a section \mathcal{H}_ξ . Since ξ is nowhere zero on S_0 , the same holds in a sufficiently small neighbourhood thereof. In this neighbourhood, the integrability of ξ , namely $\xi \wedge d\xi = 0$ implies the existence of a one-form β such that $d\xi = \xi \wedge \beta$. β is defined up to addition of $\omega\xi$ where ω is any scalar function. Thus, we can assume without loss of generality that $\beta(k) = 0$. Taking exterior derivative in $d\xi = \xi \wedge \beta$ yields $\xi \wedge d\beta = 0$, or equivalently $d\beta = \xi \wedge \gamma$ for some one-form γ . Let $\hat{\beta}$ be the pull-back on S_0 of β . From the previous considerations we know that $d\hat{\beta} = 0$ (recall that ξ is a normal one-form to S_0). Now

$$\mathbf{s}_\xi(X) = -\frac{1}{2} \langle k, \nabla_X \xi \rangle = -\frac{1}{4} d\xi(k, X) = -\frac{1}{4} (\xi \wedge \beta)(k, X) = \frac{1}{2} \beta(X) = \frac{1}{2} \hat{\beta}(X).$$

Thus $\mathbf{s}_\xi = \frac{1}{2} \hat{\beta}$ and hence \mathbf{s}_ξ is a closed one-form. The Hodge decomposition $\mathbf{s}_\xi = dQ_\xi + \mathbf{z}$ implies that $d\mathbf{z} = 0$. Since \mathbf{z} is also divergence-free, it follows that \mathbf{z} is harmonic. If S_0 admits no non-trivial harmonic one-forms (in particular if S_0 is simply connected and compact) then \mathbf{z} vanishes identically. \square

The second most relevant case where hypothesis (i) and (ii) in Proposition 1 are satisfied involve axially symmetric non-evolving horizons in four spacetime dimensions and spherical topology. We devote next section to analyze this case and to relate the stability properties of such horizons with the area-angular momentum inequalities.

5 Axially symmetric MOTS and angular momentum

We start with the following definition, which is essentially the same as the one given in [32].

Definition 3 *A MOTS (S, h) is axially symmetric if there exists a vector field $\eta \in \mathfrak{X}(S)$ with closed orbits satisfying*

- (i) $\mathcal{L}_\eta h = 0$.
- (ii) $\mathcal{L}_\eta \mathbf{s}_\ell = 0$, for some choice of basis $\{\hat{\ell}, \hat{k}\}$.
- (iii) η commutes with the stability operator L_v for some choice of normal vector v of the form (6).

Definition 4 *The angular momentum of an axially symmetric, two-dimensional MOTS S is the integral*

$$J(S) = \frac{1}{8\pi} \int_S \mathbf{s}_\ell(\eta) \boldsymbol{\eta}_S, \quad (32)$$

where the connection one-form \mathbf{s}_ℓ is defined with respect to any basis $\{\ell, k\}$.

Remark. The independence of the angular momentum with respect to the choice of basis can be found, e.g. in [29]. The argument uses only the divergence-free character of η and can be described as follows. Using $\mathbf{s}_\ell = dQ_\ell + \mathbf{z}$ yields

$$\begin{aligned} J(S) &= \frac{1}{8\pi} \int_S \mathbf{s}_\ell(\eta) \boldsymbol{\eta}_S = \frac{1}{8\pi} \int_S (dQ_\ell(\eta) + \mathbf{z}(\eta)) \boldsymbol{\eta}_S = \frac{1}{8\pi} \int_S (\operatorname{div}_h(\eta Q_\ell) + \mathbf{z}(\eta)) \boldsymbol{\eta}_S = \\ &= \frac{1}{8\pi} \int_S \mathbf{z}(\eta) \boldsymbol{\eta}_S. \end{aligned} \quad (33)$$

The last term only involves \mathbf{z} , which is independent of the choice of basis $\{\ell, k\}$.

The following theorem establishes a remarkable inequality between area and angular momentum for stable, axially symmetric MOTS in four dimensional spacetimes.

Theorem 2 (Jaramillo, Reiris, Dain, [32]) *Let $(M, g^{(4)})$ be a spacetime satisfying the dominant energy condition. Let S a two-dimensional MOTS in $(M, g^{(4)})$ and assume S to be an axially symmetric and stable with respect to the null direction $-\frac{1}{2}\hat{k}$. Then*

$$|S| \geq 8\pi |J(S)|.$$

where $|S|$ is the area of S .

A related statement for stable minimal surfaces embedded in maximal spacelike hypersurfaces in vacuum spacetimes with non-negative cosmological constant had been previously obtained by Dain and Reiris [17]. Previous to those results, area-angular momentum inequalities were obtained in the context of stationary and axially symmetric black holes by Hennig, Ansorg and Cederbaum [24]. A interesting recent review on inequalities involving area, angular momentum and mass can be found in [16] (see also [30]).

The following lemma has been used many times in the literature, see e.g. [32]. We include its proof for completeness.

Lemma 6 *Let S_0 be an axially symmetric MOTS in a spacetime $(M, g^{(n+1)})$ and chose a normal basis $\{\hat{\ell}, \hat{k}\}$ satisfying (ii) in Definition 3. Then the eigenfunction ϕ_v of the stability operator L_v , with v as in condition (iii) of Definition 3, satisfies $\mathcal{L}_\eta \phi_v = 0$.*

Moreover, if S is two-dimensional and of spherical topology, then both ϕ_v and the function $Q_{\hat{\ell}}$ in the Hodge decomposition $\mathbf{s}_{\hat{\ell}} = dQ_{\hat{\ell}} + \mathbf{z}$ satisfy $\mathbf{z}(\nabla^h \phi_v) = \mathbf{z}(\nabla^h Q_{\hat{\ell}}) = 0$.

Proof. Since L_v and \mathcal{L}_η commute, $\mathcal{L}_\eta \phi_v$ is an eigenfunction with principal eigenvalue λ_v . Hence, there exists a constant c such that $\mathcal{L}_\eta \phi_v = c\phi_v$. Integrating on S it follows

$$c \int_S \phi_v \boldsymbol{\eta}_S = \int_S \mathcal{L}_\eta(\phi_v) \boldsymbol{\eta}_S = \int_S \text{div}_h(\phi_v \boldsymbol{\eta}) \boldsymbol{\eta}_S = 0. \quad (34)$$

Thus $c = 0$ and $\mathcal{L}_\eta \phi_v = 0$. For the second part, from $\mathcal{L}_\eta \mathbf{s}_{\hat{\ell}} = 0$ it follows $\mathcal{L}_\eta Q_{\hat{\ell}} = 0$ because

$$0 = \mathcal{L}_\eta(\text{div}_h \mathbf{s}_{\hat{\ell}}) = \mathcal{L}_\eta(\Delta_h Q_{\hat{\ell}}) = \Delta_h(\mathcal{L}_\eta Q_{\hat{\ell}}) \implies \mathcal{L}_\eta Q_{\hat{\ell}} = \text{const} \implies \mathcal{L}_\eta Q_{\hat{\ell}} = 0, \quad (35)$$

where the last implication follows by integration on S as in (34). As a consequence we also have $\mathcal{L}_\eta \mathbf{z} = 0$. If, moreover, S is of spherical topology, then \mathbf{z} is the Hodge dual of the gradient of a function W and a similar argument as before implies $\mathcal{L}_\eta W = 0$. or, equivalently, that dW is orthogonal to η . S being two-dimensional, the Hodge dual \mathbf{z} of dW is tangent to η . The statements are now obvious because ϕ_v and $Q_{\hat{\ell}}$ are constant along η . \square

All sections in a totally geodesic null horizon \mathcal{H} are isometric. Moreover, the transformation law (18) implies that $\mathbf{z}[\mathbf{f}]$ is independent of the section. At first it may seem that this requires κ_ℓ to be constant, but this is not the case because one can always choose an affinely parametrized null normal ℓ_0 of \mathcal{H} . For this choice, the surface gravity is identically zero and hence (18) implies that \mathbf{s}_{ℓ_0} is independent of the section. Since, moreover, \mathbf{z} is independent of the choice of null normal basis $\{\ell, k\}$, the independence of \mathbf{z} with the section

follows. Consequently, if \mathcal{H} is three-dimensional and admits sections which are axially symmetric MOTS, then both the area $|S|$ and the angular momentum $J(S)$ are independent of the choice of axially symmetric section in \mathcal{H} . In this setting we may simply write

$$|S_{\mathcal{H}}| \geq 8\pi J(\mathcal{H}) \quad (36)$$

to refer to the area-angular momentum inequality for any of the axially symmetric section of \mathcal{H} .

Since non-evolving horizons have a preferred choice of null normal ℓ , the following definition becomes natural.

Definition 5 *Let (\mathcal{H}, ℓ) be a non-evolving horizon satisfying the topological condition (\star) . (\mathcal{H}, ℓ) is called **axially symmetric** if there exist a section S_0 of \mathcal{H} which is an axially symmetric MOTS and point (ii) in Definition (3) is satisfied by the basis $\{\hat{\ell}, \hat{k}\} = \{\ell|_{S_0}, k|_{S_0}\}$.*

We can now state and proof the following result, which gives sufficient conditions on a non-evolving horizon for the validity of (36). Its proof will be the basis for our subsequent clarification of the relationship between the argument in Hennig *et al.* and the argument in Jaramillo *et al* of their corresponding area-angular momentum inequalities.

Theorem 3 *Let (\mathcal{H}, ℓ) be an axially symmetric, non-evolving horizon in a four dimensional spacetime $(M, g^{(4)})$ satisfying the dominant energy condition (in particular (\mathcal{H}, ℓ) is an isolated horizon). Assume that \mathcal{H} is topologically $\mathbb{S}^2 \times \mathbb{R}$ with ℓ tangent to the \mathbb{R} factor and that the surface gravity κ_{ℓ} is constant and non-zero. Then, if $\int_{S_0} \kappa_{\ell} \theta_k u_{\ell} \boldsymbol{\eta}_{s_0} \leq 0$ for any section S_0 , then*

$$|S_{\mathcal{H}}| \geq 8\pi J(\mathcal{H}), \quad (37)$$

and equality occurs only if and only if the following four conditions are satisfied:

(i) *The metric h of any section of the horizon reads (in appropriate coordinates)*

$$h = |J| (1 + \cos^2 \theta) d\theta^2 + \frac{4|J| \sin^2 \theta}{1 + \cos^2 \theta} d\varphi^2. \quad (38)$$

where J is an arbitrary non-zero constant.

(ii) *There exists a section S_1 where $\text{Ein}(\ell, k) \stackrel{S_1}{=} 0$.*

(iii) *The normal connection one-form of S_1 reads*

$$s_{\ell} = -\frac{\cos \theta \sin \theta}{1 + \cos^2 \theta} d\theta + \frac{2J \sin^2 \theta}{|J| (1 + \cos^2 \theta)^2} d\varphi. \quad (39)$$

(iv) *$\int_{S_1} \theta_k [S_1] (1 + \cos^2 \theta) \boldsymbol{\eta}_{S_1} = 0$.*

Proof. Let S_0 be an axially symmetric section of \mathcal{H} . Let us start by calculating $u_\ell \|\mathbf{s}_\ell\|^2 + u_\ell \operatorname{div}_h \mathbf{s}_\ell$:

$$u_\ell \|\mathbf{s}_\ell\|_h^2 + u_\ell \operatorname{div}_h \mathbf{s}_\ell = u_\ell \|\mathbf{s}_\ell\|_h^2 + \operatorname{div}_h (u_\ell \mathbf{s}_\ell) - \mathbf{s}_\ell (\nabla^h u_\ell) = -\frac{\|\nabla^h u_\ell\|_h^2}{4u_\ell} + u_\ell \|\mathbf{z}\|_h^2 + \operatorname{div}_h (u_\ell \mathbf{s}_\ell),$$

where in the last equality we have used $\mathbf{s}_\ell = \frac{du_\ell}{2u_\ell} + \mathbf{z}$ and the orthogonality between \mathbf{z} and du_ℓ . Inserting this in expression (15) (with $\mathcal{N}^\ell = 0$) and integrating on S_0 it follows

$$\int_{S_0} \left(\frac{\|\nabla^h u_\ell\|_h^2}{2u_\ell} + u_\ell \operatorname{Scal}_h \right) \boldsymbol{\eta}_{S_0} = \int_{S_0} (u_\ell \operatorname{Ein}(\ell, k) - \kappa_\ell \theta_k u_\ell + 2u_\ell \|\mathbf{z}\|_h^2) \boldsymbol{\eta}_{S_0}. \quad (40)$$

Under the conditions of the theorem, the first two terms in the right-hand side are non-negative. Discarding them yields

$$\int_{S_0} \left(\frac{\|\nabla^h u_\ell\|_h^2}{2u_\ell} + u_\ell \operatorname{Scal}_h \right) \boldsymbol{\eta}_{S_0} \geq \int_{S_0} 2u_\ell \|\mathbf{z}\|_h^2 \boldsymbol{\eta}_{S_0}. \quad (41)$$

The area-angular momentum inequality is proved in [32] by choosing a coordinate system on S_0 in which the metric h reads (this form of the metric was introduced in [8], a detailed proof of existence of the coordinate system appears in [17])

$$h = e^\sigma (e^{2q} d\theta^2 + \sin^2 \theta d\varphi^2), \quad (42)$$

where σ, q are functions of θ satisfying $q + \sigma = c$, where c is constant (see expression (13) in [32]). The inequality $|S_0| \geq 8\pi J(S_0)$ is proved in that paper as a consequence of the inequality

$$\int_{S_0} \left(\|\nabla^h \alpha\|_h^2 + \frac{\alpha^2}{2} \operatorname{Scal}_h \right) \boldsymbol{\eta}_{S_0} \geq \int_{S_0} \alpha^2 \|\mathbf{z}\|^2 \boldsymbol{\eta}_{S_0} \quad (43)$$

where α is an arbitrary function which is then chosen to be related to the metric h by $\alpha = e^c e^{-\sigma/2}$. This inequality, in turn, follows from the stability of the MOTS along $-\frac{1}{2}\hat{k}$. The close relationship between (41) and (43) is apparent. The freedom in u_ℓ in (41) comes from the freedom in choosing the section S_0 . Given the transformation law (29) for u_ℓ , we can choose the graph function f so that

$$u_\ell[f] = u_\ell e^{2\kappa_\ell f} = \alpha^2 = e^{2c} e^{-\sigma}. \quad (44)$$

The section $S_1 \stackrel{\text{def}}{=} S[f]$ is still axially symmetric. The argument in [32] applied to S_1 proves $|S_1| \geq 8\pi J(S_1)$. Since the inequality is independent of the (axially symmetric) section, (37) follows.

For the equality case, it is proved in [32], [1] that equality in (43) occurs if and only if the following two conditions hold: (i) the geometry on S_1 (and hence of any section of the horizon) is isometric to the extreme Kerr throat geometry, given explicitly by (38) and (ii) the one-form \mathbf{z} takes the form

$$\mathbf{z} = \frac{2J \sin^2 \theta}{|J| (1 + \cos^2 \theta)^2} d\varphi. \quad (45)$$

In our setting we have, in addition, discarded two non-negative terms in (41). This forces $\text{Ein}(\ell, k) \stackrel{S_1}{=} 0$ and $\int_{S_1} \kappa_\ell \theta_k [S_1] u_\ell [S_1] \boldsymbol{\eta}_{S_1} = 0$. Moreover, since $u_\ell[f] = e^{2c} e^{-\sigma}$ it follows from (38) and (42) that

$$u_\ell[S_1] = |J| (1 + \cos^2 \theta).$$

Inserting this function and (45) into $s_\ell = \frac{du_\ell[S_1]}{2u_\ell[S_1]} + \boldsymbol{z}$ gives (39). Condition (iv) follows directly from $\int_{S_1} \kappa_\ell \theta_k [S_1] u_\ell [S_1] \boldsymbol{\eta}_{S_1} = 0$ and the explicit form of u_ℓ . \square

Remark. As mentioned above, inequality (43) is proven in [32] from the stability of the MOTS by using direct estimates. An alternative derivation can be obtained from the Rayleigh-Ritz type characterization of the principal eigenvalue obtained in expression (16) in [3] and using the fact that, whenever the function u in that paper is axially symmetric then $\omega[u] = 0$.

Combining this theorem with Proposition 5 yields the following Corollary.

Corollary 5 *Let (\mathcal{H}, ℓ) be an axially symmetric, non-evolving horizon in a four dimensional spacetime $(M, g^{(4)})$ satisfying the dominant energy condition. Assume that \mathcal{H} is topologically $\mathbb{S}^2 \times \mathbb{R}$ with ℓ tangent to the \mathbb{R} factor and that the surface gravity κ_ℓ is constant and non-zero. If \mathcal{H} is stable then*

$$|S_{\mathcal{H}}| \geq 8\pi J(\mathcal{H}),$$

and equality occurs if and only if the following condition hold

- (i) *The horizon is marginally stable.*
- (ii) *The metric h of any section of the horizon reads as in (38).*
- (iii) *There exists a minimal section S_1 (i.e. a section satisfying $\theta_k[S_1] = 0$).*
- (iv) *The Einstein tensor satisfies $\text{Ein}(\ell, k) \stackrel{S_1}{=} 0$.*
- (v) *The normal connection one-form of S_1 reads as in (39).*

We are now in a position where we can explain in which sense Theorem 3 clarifies the relationship between the area-angular momentum inequality of Hennig, Ansorg and Cederbaum [24] and the area-angular momentum inequality obtained by Jaramillo, Reiris and Dain [32].

The setup in [24] deals with stationary and axially symmetric black hole spacetimes which are vacuum in a neighbourhood of the event horizon. The whole argument is performed in adapted coordinates where the metric takes the following form.

$$\begin{aligned} ds^2 = & \left(\frac{a}{\hat{u}} + \hat{u} b^2 \sin^2 \theta \right) dR^2 - \frac{\Delta}{\hat{u}} d\tilde{t}^2 + \hat{u} \sin^2 \theta (d\tilde{\varphi} - \omega d\tilde{t})^2 + \hat{\mu} d\theta^2 \\ & + 2 \left(\frac{T}{\hat{u}} + \omega \hat{u} b \sin^2 \theta \right) dR d\tilde{t} - 2 \hat{u} b \sin^2 \theta dR d\tilde{\varphi} \end{aligned}$$

where $\Delta = 4(R^2 - r_h^2)$, r_h is a positive constant which defines the location of the horizon \mathcal{H} (at $R = r_h$). The functions $a, b, \hat{u}, \hat{\mu}, \omega$ depend on $\{R, \theta\}$ and satisfy $\hat{u} > 0$, $\hat{\mu} > 0$, $\omega|_{\mathcal{H}} = \omega_h$ constant and $\frac{2r_h}{\sqrt{\hat{\mu}\hat{u}}} = \kappa > 0$ constant. In fact, κ is precisely the surface gravity of the Killing vector for which \mathcal{H} is a Killing horizon, namely $\xi = \partial_{\tilde{t}} + \omega_h \partial_{\tilde{\varphi}}$. The function T depends on R alone and satisfies $T(R = r_h) = \frac{4r_h}{\kappa}$ and $\frac{dT}{dR}|_{R=r_h} < 0$. The sections of the horizon considered in that paper are $\mathcal{S} \stackrel{\text{def}}{=} \{\tilde{t} = \text{const.}\}$. The key condition imposed by Hennig *et al* is the *subextremality* of the horizon, namely the existence of inward variations from \mathcal{S} which strictly decrease the null expansion θ_k at every point (this is known to be equivalent to the strict stability of the horizon, see Proposition 5.1 in [3]). In fact, the authors do not quite need this condition, but a weaker condition stated in Lemma 3.1 in [24], namely

$$\int_0^\pi \frac{\partial(\hat{\mu}\hat{u})}{\partial R} \Big|_{R=r_h} \sin \theta d\theta > 0. \quad (46)$$

This inequality is sufficient to prove $|\mathcal{S}| > 8\pi J(S)$ [24]. Now, we can understand this result in the light of Theorem 3. It is matter of straightforward calculation to determine the induced metric h , normal connection one-form $\mathbf{s}_\xi[\mathcal{S}]$ and null expansion $\theta_k[\mathcal{S}]$ of \mathcal{S} . Letting $\tilde{u} \stackrel{\text{def}}{=} \hat{u}|_{r=r_h}$, the result is

$$\begin{aligned} h &= \frac{4r_h^2}{\kappa^2 \tilde{u}} d\theta^2 + \tilde{u} \sin^2 \theta d\tilde{\varphi}^2, \\ \mathbf{s}_\xi[\mathcal{S}] &= -\frac{1}{2\tilde{u}} \frac{\partial \tilde{u}}{\partial \theta} d\theta + \frac{\kappa \tilde{u}^2 \sin^2 \theta}{8r_h} \frac{\partial \omega}{\partial R} \Big|_{R=r_h} d\tilde{\varphi}, \\ \theta_k[\mathcal{S}] &= -\frac{\kappa^3 \tilde{u}}{16r_h^3} \frac{\partial(\hat{u}\hat{\mu})}{\partial R} \Big|_{R=r_h}. \end{aligned}$$

According to the definition of u_ℓ we conclude that $u_\ell[\mathcal{S}] = a_0 \tilde{u}^{-1}$ where a_0 is an arbitrary positive constant. We observe, first of all, that the metric h has the form (42) with $e^\sigma = \tilde{u}$, $e^q = \frac{2r_h}{\kappa \tilde{u}}$ and $e^c = \frac{2r_h}{\kappa}$. Moreover, condition (44) is automatically satisfied if we choose $a_0 = \frac{4r_h^2}{\kappa^2}$. Consequently, we have

$$-\int_{\mathcal{S}} \kappa_\ell u_\ell[\mathcal{S}] \theta_k[\mathcal{S}] \boldsymbol{\eta} \mathbf{s} = \int_{\mathcal{S}} \frac{\kappa^2}{4r_h} \frac{\partial(\hat{u}\hat{\mu})}{\partial R} \Big|_{R=r_h} \sin \theta d\theta d\tilde{\varphi} = \frac{\pi \kappa^2}{2r_h} \int_0^\pi \frac{\partial(\hat{u}\hat{\mu})}{\partial R} \Big|_{R=r_h} \sin \theta d\theta \quad (47)$$

so that the integral condition (46) is precisely the same as the requirement $\int_{\mathcal{S}} \kappa_\ell \theta_k[\mathcal{S}] u_\ell[\mathcal{S}] \boldsymbol{\eta} \mathbf{s} < 0$, which is directly related to the main hypothesis of Theorem 3. As we have seen along the proof of this theorem, the key inequality behind $|\mathcal{S}| \geq 8\pi J(S)$ is (43). In one case, this inequality follows from the stability of the MOTS and a suitable choice of coordinates in S and a choice of α expressed in terms of the metric. In the other case, the inequality follows from $\int_{S_0} \kappa_\ell \theta_k u_\ell \boldsymbol{\eta} \mathbf{s}_0 \leq 0$ after exploiting the freedom in choosing the section of the horizon. It is remarkable that the coordinate system used in [24] has the property that the sections $\tilde{t} = \text{const.}$ are precisely the sections $S[f]$ for which the integral condition $-\int_{S[f]} \kappa u_\ell[f] \theta_k[f] \boldsymbol{\eta} \mathbf{s}[f] > 0$ becomes exactly the integral in the left-hand side of (43).

As mentioned in the Introduction, the relationship between the proof in [24] and the proof of the area-angular momentum inequality for stable minimal surfaces lying in maximal vacuum initial data sets [17] has been clarified recently in [15] by showing in an appropriate coordinate system that inequality (46) is precisely the strict version of (43). The latter is, in this setting, a consequence of the Rayleigh-Ritz quotient characterization of the principal eigenvalue of the stability operator for minimal surfaces. The clarification we have obtained in this paper provides, in addition, a clear geometric interpretation of (46) in terms of the geometry of the Killing horizon.

A final remark is in order. In [15] the final step for the existence of singularities in two-Kerr black hole spacetime has been achieved. In [37], [26] and [27] it was proved that the double Kerr solution of Kramer and Neugebauer [33] is the only possible candidate for a stationary and axially symmetric asymptotically flat black hole with an event horizon of two connected components. Moreover, the authors also prove that either if both components are degenerate or if the strict inequality $|S| > 8\pi J(S)$ holds on each non-degenerate component, then there must exist a conical singularity in the portion of the axis of symmetry lying between each connected component of the event horizon. The proof was based on explicit formulae obtained previously by Manko and Ruiz [36] where existence of singularities was shown under positivity of the Komar masses of each black hole constituent. In [15] the stability (in the sense of MOTS) of each connected component of the event horizon is proved as a consequence of existence of an outermost MOTS in spacelike asymptotically flat slices [4, 18, 2]. Stability only implies $|S| \geq 8\pi J(S)$ so one might think that the results by Hennig and Neugebauer are, by themselves, not quite sufficient to finish the proof. In [15] this issue is dealt with by mentioning (with no explicit proof) that the arguments in by Hennig and Neugebauer can be extended to cover the case $|S| = 8\pi J(S)$ on non-degenerate components.

In view of the results in this paper, an alternative argument to exclude the equality case $|S| = 8\pi J(S)$ in non-degenerate components is to show that any of the five conditions (i)-(v) in Corollary 5 is not satisfied in the double Kerr spacetime. The coordinate transformation $\cos \theta = a_0(\zeta - \zeta_0)$, with a_0, ζ_0 constants brings the metric (38) into the form

$$h = \frac{a_1^2}{F(\zeta)} d\zeta^2 + a_2^2 F(\zeta) d\varphi^2, \quad (48)$$

with $a_1 \stackrel{\text{def}}{=} a_0 \sqrt{|J|}$, $a_2 \stackrel{\text{def}}{=} 2\sqrt{|J|}$, and

$$F(\zeta) \stackrel{\text{def}}{=} \frac{1 - a_0^2 (\zeta - \zeta_0)^2}{1 + a_0^2 (\zeta - \zeta_0)^2}.$$

In fact, it is straightforward to check that $\cos \theta = a_0(\zeta - \zeta_0)$ is the most general transformation that brings (38) into the form (48). On the other hand the geometry of a non-degenerate connected component of the “event horizon” of the double Kerr solution is

$$\tilde{h} = \frac{a_1^2}{\tilde{F}(\zeta)} d\zeta^2 + a_2^2 \tilde{F}(\zeta) d\varphi^2, \quad (49)$$

where $-\tilde{F}(\zeta)$ is the real part of the Ernst potential associated to the asymptotically timelike Killing vector $\hat{\xi}$. Expression (49) follows by setting $\{\rho = 0, t = 0\}$ in the spacetime metric written in Weyl-Papapetrou coordinates $\{t, \varphi, \rho, \zeta\}$ as written e.g. in formula (1) in [27] and using the facts that the functions k and a in that metric are constant along the horizon. Now, the Ernst potential associated to the Killing vector $\hat{\xi}$ for the double Kerr solution is well-known. Its explicit form is given, for instance, in expression (34.94) in [39]. Restricting to $\rho = 0$, $\zeta \in (K_4, K_3)$ ($K_4 < K_3 < K_2 \leq K_1$ are constants) which corresponds to a non-degenerate component of the horizon, and taking the real part yields

$$\tilde{F}(\zeta) = \frac{(K_1 - \zeta)(K_2 - \zeta)(K_3 - \zeta)(\zeta - K_4)}{P_4(\zeta)},$$

where $P_4(\zeta)$ is a fourth-order polynomial which does not vanish anywhere in the interval $\zeta \in (K_4, K_3)$. It is obvious that \tilde{h} is not isometric to h in (48), so we are not in the equality case of the area-angular momentum inequality, and non-degenerate horizons in the double Kerr spacetime necessarily satisfy the strict inequality $|S| > 8\pi J(S)$.

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